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SMALL DEFORMATIONS SUPERPOSED ON LARGE DEFORMATIONS IN MATERIALS WITH FADING MEMORY

by

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Erratum:

The letter "v", used as subscript or a superscript, should be replaced by "ν" throughout.

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SMALL DEFORMATIONS SUPERPOSED ON LARGE DEFORMATIONS IN MATERIALS WITH FADING MEMORY*

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1. Introduction

The non-linear stress-deformation relation for materials with memory has been discussed by Green and Rivlin [1]**** and Green, Rivlin and Spencer [2]. In the present paper we derive the restricted form applicable to small dynamic deformations superposed on large static deformations. Although this can be done by making an appropriate linearization of the final result of the non-linear theory, we avoid some of the complexities of that theory by introducing the linearization at an earlier stage. The results obtained will apply in particular to problems of wave propagation in finitely deformed viscoelastic bodies.

It is assumed that the stress depends on the deformation history, and that the properties of the material do not change with time. The stress components are then hereditary functionals of the deformation gradients. Because rigid rotation of a deformed body has no effect on stress, other than to rotate the stress field, and since the stress-deformation relation must embody this fact, the relation is of a certain restricted form. This form was derived by Green and Rivlin [1] by using invariant-

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**** Numbers in square brackets refer to the References at the end of this paper.

theoretic methods, and subsequently Noll [3] derived an equivalent form. Noll's derivation is simpler than that of Green and Rivlin in that it does not make use of invariant theory, but the form he derives is not suitable for the purposes of the present paper. In Section 3 we take the opportunity to present a derivation which combines some of the simplicity of Noll's method with the explicitness of Green and Rivlin's result.

When the deformation consists of an infinitesimal displacement field superimposed on a finite deformation, the stress-deformation relation can be linearized with respect to the gradients of the infinitesimal displacement field (Section 4). In a material with fading memory, if the finite deformation is held fixed long enough so that the original deforming process has no further effect, the non-linear functionals specifying the stress can be expressed in terms of non-linear functions and linear functionals (Sections 5 and 6).

The functions appearing in the linearized formulation are subject to restrictions in form if the material has any symmetries in its undeformed state (Section 7). In the case of isotropic materials (Section 8), these restrictions are such that the constitutive equation can be brought into a simpler form involving the classical strain tensor (Section 9).

2. Materials with memory and hereditary functionals

Consider a body in its undeformed state. Let the coordinates of a generic particle be X_i, in a fixed Cartesian coordinate system x. If the body is deformed as time progresses, then the particle which was originally at X_i moves to a new position x_i at time t, and the deformation may be specified by giving x_i as a function of X_p and t:

$$x_i = x_i(X_p, t) \ . \tag{2.1}$$

The derivatives $\partial x_i/\partial X_p$, called deformation gradients, may be used as measures of the deformation in the neighborhood of the particle X_p.

Suppose that the stress at a given particle depends on the deformation gradients at that particle, not only at the instant t considered but also at all previous times. The stress components σ_{ij} at time t are then functionals of the deformation gradients at the particle considered. To express this idea, we write

$$\sigma_{ij}(t) = f_{ij}[dx_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q;\ t] \ . \tag{2.2}$$

Explicit dependence of the functionals on t indicates that the intrinsic properties of the material may change as time progresses, whether the material is subjected to a deformation or not. This possibility will immediately be rejected, however, for we assume that the material has the following property: if two deformation histories differ only by a time shift, then the corresponding stress histories differ only by that same time shift. It is easy to see, and correspondingly easy to show

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$$\sigma_{ij}(t) = f_{ij}[\partial x_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q]. \qquad (2.3)$$

The functionals are then said to be <u>hereditary</u>.

3. Rotation of the physical system

Not every relation of the form (2.3) is physically meaningful. When an arbitrary rigid rotation is superimposed on a given deformation history, the stress field undergoes an equal rigid rotation. The form of the constitutive equation must be such that this condition is satisfied.

Let $\bar{x}_i$ be a deformation history obtained by superimposing a rigid rotation on the deformation $x_i(t)$, so that

$$\bar{x}_i\,(X_p,t) = a_{ij}(t)x_j(X_p,t)\ . \tag{3.1}$$

Here the coefficients $a_{ij}(t)$ must satisfy the orthogonality conditions

$$a_{ik}(t)a_{jk}(t) = a_{ki}(t)a_{kj}(t) = \delta_{ij}\ , \qquad |\,a_{ij}(t)\,| = 1. \tag{3.2}$$

From Eq. (3.1), the deformation gradients for the new deformation are

$$\partial\bar{x}_i(t)/\partial X_p = a_{ij}(t)\partial x_j(t)/\partial X_p\ . \tag{3.3}$$

Letting $\bar{\sigma}_{ij}(t)$ denote the stress components associated with the deformation $\bar{x}_i(t)$, Eq. (2.3) implies that

$$\bar{\sigma}_{ij}(t) = f_{ij}[\partial\bar{x}_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q]\ . \tag{3.4}$$

The condition to be satisfied is that the components $\bar{\sigma}_{ij}(t)$ are related to the components $\sigma_{ij}(t)$ by the rotation $a_{ij}(t)$. Thus,

$$\bar{\sigma}_{ij}(t) = a_{ik}(t)a_{j\ell}(t)\sigma_{k\ell}(t)\ . \tag{3.5}$$

By using Eqs. (2.3) and (3.4) in Eq. (3.5), and making use of Eqs. (3.2) and (3.3), we obtain

$$f_{ij}[\partial x_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q] \tag{3.6}$$
$$= a_{ki}(t)a_{\ell j}(t)f_{k\ell}[a_{pr}(t - \tau)\underset{\tau=0}{\overset{\infty}{\partial x_r}}(t - \tau)/\partial X_q].$$

The functionals f_{ij} must be of such form that Eq.(3.6) is satisfied no matter what the choice of $a_{ij}(t)$ may be, subject to the restrictions (3.2).

We may in particular let $a_{ij}(t)$ be the j-component of vector number i in an orthogonal system, formed from the three vectors $\partial x_j/\partial X_1$, $\partial x_j/\partial X_2$, and $\partial x_j/\partial X_3$ by the Schmidt orthogonalization process. In the first step of this process, $a_{1j}(t)$ is constructed by dividing $\partial x_j(t)/\partial X_1$ by its length. We thus obtain

$$a_{1j}(t) = [G_{11}(t)]^{-\frac{1}{2}} \partial x_j(t)/\partial X_1 \; . \tag{3.7}$$

Here $G_{11}(t)$ is one of the set of inner products $G_{pq}(t)$, called strain components, which are defined by

$$G_{pq}(t) = \frac{\partial x_i(t)}{\partial X_p} \frac{\partial x_i(t)}{\partial X_q} \; . \tag{3.8}$$

Next, we construct the unit vector in the plane of $\partial x_j/\partial X_1$ and $\partial x_j/\partial X_2$, perpendicular to $\partial x_j/\partial X_1$ and forming a positive inner product with $\partial x_j/\partial X_2$. Using the notation $\Delta_{pq}(t)$ for the cofactor of $G_{pq}(t)$ in the determinant $G(t) = |G_{pq}(t)|$, this vector may be written

$$a_{2j}(t) = [G_{11}(t)\Delta_{33}(t)]^{-\frac{1}{2}}[-G_{12}(t)\partial x_j(t)/\partial X_1 + G_{11}(t)\partial x_j(t)/\partial X_2] \; . \qquad (3.9)$$

Finally, $a_{3j}(t)$ is taken to be the unit vector perpendicular to $\partial x_j/\partial X_1$ and $\partial x_j/\partial X_2$ whose inner product with $\partial x_j/\partial X_3$ is positive. Thus,

$$a_{3j}(t) = [\Delta_{33}(t)G(t)]^{-\frac{1}{2}} [\Delta_{13}(t) \frac{\partial x_j(t)}{\partial X_1} + \Delta_{23}(t) \frac{\partial x_j(t)}{\partial X_2} + \Delta_{33}(t) \frac{\partial x_j(t)}{\partial X_3}] \; . \qquad (3.10)$$

We note that $G_{11}(t)$, $\Delta_{33}(t)$, and $G(t)$ are positive in any deformation possible in a real material. Equations (3.7), (3.9) and (3.10) may be abbreviated as

$$a_{ij}(t) = C_{ik}(t)\partial x_j(t)/\partial X_k \; , \qquad (3.11)$$

where the coefficients $C_{ik}(t)$ are functions of the strain components $G_{pq}(t)$.

Since the coefficients $a_{ij}(t)$ constructed in the manner described above satisfy the restrictions (3.2), Eq. (3.6) must be satisfied for this choice of $a_{ij}(t)$. Thus, if Eq. (3.11) is used in Eq. (3.6), and Eq. (3.8) is used to simplify the expression for the functional argument, we obtain

$$\begin{aligned} f_{ij}[\partial x_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q] &= C_{km}(t)\frac{\partial x_i(t)}{\partial X_m} C_{\ell n}(t)\frac{\partial x_j(t)}{\partial X_n} f_{k\ell}[C_{pr}(t-\tau)\underset{\tau=0}{\overset{\infty}{G_{rq}}}(t-\tau)] \\ &= \frac{\partial x_i(t)}{\partial X_m} \frac{\partial x_j(t)}{\partial X_n} F_{mn}[\underset{\tau=0}{\overset{\infty}{G_{pq}}}(t-\tau)] \; , \quad \text{(say)}. \end{aligned} \qquad (3.12)$$

The second member of this equality is of the form shown in the last member because the coefficients $C_{ij}(t)$ are functions of the strain components $G_{pq}(t)$.

It is thus seen that if Eq. (3.6) is satisfied, the functionals f_{ij} can be written in the form (3.12). It is easy to show that, conversely, if the functionals are of the form (3.12), then Eq. (3.6) is satisfied for all orthogonal transformations. The constitutive equation is therefore of the restricted form

$$\sigma_{ij}(t) = \frac{\partial x_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_\ell} \; F_{k\ell}[G_{pq}(t - \tau)]_{\tau=0}^{\infty} \; . \qquad (3.13)$$

We note that since the stress tensor is symmetric, $F_{k\ell} = F_{\ell k}$.

4. Small deformations superimposed on large deformations

Consider a deformation obtained by superimposing a displacement field $\varepsilon u_i(X_p,t)$ on the basic deformation $x_i = x_i(X_p,t)$:

$$x_i^*(X_p,t) = x_i(X_p,t) + \varepsilon u_i(X_p,t). \tag{4.1}$$

The stress components corresponding to this deformation are, from Eq. (3.13),

$$\sigma_{ij}(t) = \frac{\partial x_i^*(t)}{\partial X_k}\frac{\partial x_j^*(t)}{\partial X_\ell}\, \underset{\tau=0}{\overset{\infty}{F_{k\ell}}}[G_{pq}^*(t-\tau)], \tag{4.2}$$

where the strain components $G_{pq}^*(t)$ are those obtained by replacing $x_i(t)$ by $x_i^*(t)$ in Eq. (3.8). We wish to obtain the linearized form of Eq. (4.2), valid for small enough values of ε .

By using Eq. (4.1) and neglecting terms of order ε^2, we obtain

$$\frac{\partial x_i^*(t)}{\partial X_k}\frac{\partial x_j^*(t)}{\partial X_\ell} = \frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell} + \varepsilon\left[\frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell} + \frac{\partial x_i(t)}{\partial X_k}\frac{\partial u_j(t)}{\partial X_\ell}\right]. \tag{4.3}$$

The strain components $G_{pq}^*(t)$ appropriate to the new history can be found by setting i=j in Eq. (4.3). Thus,

$$G_{k\ell}^*(t) = G_{k\ell}(t) + 2\varepsilon E_{k\ell}(t) . \tag{4.4}$$

Here the functions $G_{k\ell}(t)$ are the strain components for the basic deformation $x_i(t)$, and the functions $E_{k\ell}(t)$ are defined by

$$2E_{k\ell}(t) = \frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_i(t)}{\partial X_\ell} + \frac{\partial x_i(t)}{\partial X_k}\frac{\partial u_i(t)}{\partial X_\ell} . \tag{4.5}$$

We now assume that the functional $F_{k\ell}$ is particularly dependent on the value of $G^*_{pq}(t-\tau)$ and its time derivatives at $\tau = 0$, and is a continuous functional of $G^*_{pq}(t-\tau)$ over the range $0 < \tau < \infty$. Using the notation

$$G^{*(0)}_{pq} = G^*_{pq}(t) \ , \qquad G^{*(\mu)}_{pq} = d^\mu G^*_{pq}(t)/dt^\mu \ , \tag{4.6}$$

we suppose that $F_{k\ell}$ may be expressed as the sum of a number of terms, each of which is the product of a differentiable function f_v, say, of $G^{*(\mu)}_{pq}$ $(\mu=0,1,2,\ldots)$, and a continuous functional Φ_v, say, of $G^*_{pq}(t-\tau)$ over the range $0 < \tau < \infty$. Then,

$$F_{k\ell}[G^*_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}}] = \sum_v f_v(G^{*(\mu)}_{pq})\Phi_v[G^*_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}}] \ . \tag{4.7}$$

Using Taylor's theorem, we have, with (4.4) and the neglect of terms of order higher than the first in ε,

$$f_v(G^{*(\mu)}_{pq}) = f_v(G^{(\mu)}_{pq}) + 2\varepsilon \sum_\mu E^{(\mu)}_{pq} \, \partial f_v/\partial G^{*(\mu)}_{pq}\Big|_{\tau=0} \ . \tag{4.8}$$

In (4.8) we use the notation

$$E^{(0)}_{pq} = E_{pq}(t) \ , \qquad E^{(\mu)}_{pq} = d^\mu E_{pq}(t)/dt^\mu \ . \tag{4.9}$$

By using the integral representation for continuous functionals, it is seen that neglecting terms of order ε^2, the functional Φ_v may be expressed in the form

$$\begin{aligned}\Phi_v[G^*_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}}] &= \Phi_v[G_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}}] \\ &\quad + \varepsilon \int_0^\infty \Phi^{(v)}_{rs}[G_{pq}\overset{\infty}{\underset{\xi=0}{(t-\xi)}};\ \tau]E_{rs}(t-\tau)d\tau \ .\end{aligned} \tag{4.10}$$

By using (4.8) and (4.10) in (4.7), and again neglecting terms of order ε^2, we see that $F_{k\ell}$ is expressible in the form

$$F_{k\ell}[\overset{\infty}{\underset{\tau=0}{G^*_{pq}}}(t-\tau)] = F_{k\ell}[\overset{\infty}{\underset{\tau=0}{G_{pq}}}(t-\tau)] + \varepsilon \sum_{v} A^{(v)}_{k\ell rs}[\overset{\infty}{\underset{\tau=0}{G_{pq}}}(t-\tau);\, G^{(\mu)}_{pq}]E^{(v)}_{rs}(t) + \varepsilon \int_0^\infty K_{k\ell rs}[\overset{\infty}{\underset{\xi=0}{G_{pq}}}(t-\xi);\, G^{(\mu)}_{pq},\, \tau]E_{rs}(t-\tau)d\tau. \quad (4.11)$$

The constitutive equation for small deformations superimposed on large deformations is now obtained by using Eqs. (4.3) and (4.11) in (4.2), again neglecting terms of order ε^2. Since ε was introduced only to facilitate the linearization process, it will be omitted after this step; i.e. we take $\varepsilon=1$ with the stipulation that the displacement gradients and their derivatives are small. The linearized equation is then

$$\begin{aligned}\sigma_{ij}(t) = {} & \Big[\frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell} + \frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell} + \frac{\partial x_i(t)}{\partial X_k}\frac{\partial u_j(t)}{\partial X_\ell}\Big]F_{k\ell}[\overset{\infty}{\underset{\tau=0}{G_{pq}}}(t-\tau)] \\ & + \frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell}\sum_{v} A^{(v)}_{k\ell rs}[\overset{\infty}{\underset{\tau=0}{G_{pq}}}(t-\tau);\, G^{(\mu)}_{pq}]E^{(v)}_{rs}(t) \\ & + \frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_\ell}\int_0^\infty K_{k\ell rs}[\overset{\infty}{\underset{\xi=0}{G_{pq}}}(t-\xi);\, G^{(\mu)}_{pq},\, \tau]E_{rs}(t-\tau)d\tau\,. \end{aligned} \quad (4.12)$$

Since the stress tensor is symmetric, then $F_{k\ell}=F_{\ell k}$, $A^{(v)}_{k\ell rs}=A^{(v)}_{\ell krs}$, and $K_{k\ell rs}=K_{\ell krs}$. Since $E^{(v)}_{rs}=E^{(v)}_{sr}$, there is no loss of generality in taking $A^{(v)}_{k\ell rs}=A^{(v)}_{k\ell sr}$ and $K_{k\ell rs}=K_{k\ell sr}$ as well.

5. Materials with fading memory

If the basic deformation $x_i(X_p,t)$ were time-independent, the functionals F_{ij}, $A^{(v)}_{ijk\ell}$, and $K_{ijk\ell}$ in Eq. (4.12) would reduce to ordinary functions of the constant strain components G_{pq}. The basic deformation is, of course, necessarily time-dependent, because it must involve a transition from the undeformed state to whatever fixed state is contemplated. However, the same simplification is achieved if a material with fading memory is held in a fixed state of deformation for so long that the original deforming process is forgotten.

Let the basic deformation $x_i(X_p,t)$ in Eq. (4.1) be one in which the body remains in a fixed state $x_i(X_p)$ after time $t=0$; i.e.

$$x_i(X_p,t) = x_i(X_p), \quad t > 0. \tag{5.1}$$

Using the notation $G_{pq}(t)$ for the strain histories, as before, let G_{pq} denote the constant strains associated with the fixed deformation $x_i(X_p)$. Since $G_{pq}(t) = G_{pq}$ for $t>0$, then the derivatives $G^{(\mu)}_{pq}$ $(\mu \neq 0)$ in Eq. (4.12) all vanish.

We assume that after a sufficient amount of time has elapsed, the deformation history before $t=0$ has an arbitrarily small effect on the stress. This assumption requires that the following condition be satisfied by the functionals $F_{k\ell}$ which appear in Eq. (4.2):

$$\lim_{t \to \infty} \{ F_{ij}[G_{pq} \overset{\infty}{\underset{\tau=0}{(t-\tau)}} + 2E_{pq}(t-\tau)] - F_{ij}[G_{pq} + 2E_{pq} \overset{\infty}{\underset{\tau=0}{(t-\tau)}}] \} = 0. \tag{5.2}$$

As a particular case of Eq. (5.2), we have

$$\lim_{t \to \infty} F_{ij}[G_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}}] = F_{ij}[\overset{\infty}{\underset{\tau=0}{G_{pq}}}] = F_{ij}(G_{pq}), \text{ (say)} . \quad (5.3)$$

As indicated, the functionals reduce to functions of the constant values G_{pq} in this case. By considering the linearized form of Eq. (5.2), i.e. the form corresponding to Eq. (4.12), it can be shown that the functionals $A^{(v)}_{ijk\ell}$ and $K_{ijk\ell}$ also reduce to functions of G_{pq} at large times.

When the deformation has been held fixed long enough that the history before t=0 has no perceptible effect, the functionals F_{ij}, $A^{(v)}_{ijk\ell}$, and $K_{ijk\ell}$ may be replaced by their limiting forms, which are functions. The constitutive equation (4.12) then assumes the form

$$\sigma_{ij}(t) = [\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_\ell} + \frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j}{\partial X_\ell} + \frac{\partial x_i}{\partial X_k}\frac{\partial u_j(t)}{\partial X_\ell}]F_{k\ell}(G_{pq})$$

$$+ \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_\ell}[\sum_v A^{(v)}_{k\ell rs}(G_{pq})E^{(v)}_{rs}(t)$$

$$+ \int_0^\infty K_{k\ell rs}(G_{pq},\tau)E_{rs}(t-\tau)d\tau] . \quad (5.4)$$

Here x_i denotes the basic steady deformation, with associated strain components G_{pq}.

In passing, we note that if u_i is steady, then the combination x_i+u_i can be regarded as a new steady deformation with no unsteady part superimposed. The stress given by Eq. (5.4) must be the same, whichever way the deformation is regarded. From this consideration, we find after some manipulation that the following relation must be satisfied:

$$A^{(0)}_{ijk\ell}(G_{pq}) + \int_0^\infty K_{ijk\ell}(G_{pq},\tau)d\tau = 2\,\frac{\partial F_{ij}(G_{pq})}{\partial G_{k\ell}}\,. \tag{5.5}$$

In the remainder of this paper we will consider only those materials and deformations for which Eq. (5.4) is valid.

6. An alternative approach to materials with fading memory

In Sections 2 to 5 we have developed the constitutive equation for small dynamic deformations superimposed on large steady deformations in materials with fading memory. This was done by specializing the constitutive equation for materials with an unspecified type of memory. An alternative approach is possible, in which it is assumed initially that the deformation consists of a steady deformation $x_i(X_p)$ with a displacement field $u_i(X_p,t)$ superimposed on it. It is then assumed that the stress components depend on the deformation gradients $\partial x_p/\partial X_q$ and the history of the displacement gradients $\partial u_p(t)/\partial X_q$, so that

$$\sigma_{ij}(t) = f_{ij}[\partial u_p(t-\tau)/\partial X_q \Big|_{\tau=0}^{\infty} ; \partial x_p/\partial X_q] . \tag{6.1}$$

To find the restricted forms of f_{ij} which will satisfy the rotation invariance principle, we consider a new total deformation $\bar{x}_i(t) = \bar{x}_i + \bar{u}_i(t)$ related to the original total deformation $x_i(t) = x_i + u_i(t)$ by a rigid rotation, i.e. by Eq. (3.1). The stress components corresponding to the new deformation are given by

$$\bar{\sigma}_{ij}(t) = f_{ij}[\partial \bar{u}_p(t-\tau)/\partial X_q ; \partial \bar{x}_p/\partial X_q] . \tag{6.2}$$

The new stress components $\bar{\sigma}_{ij}(t)$ must then be related to the components $\sigma_{ij}(t)$ by the transformation (3.5). By using Eqs. (6.1) and (6.2) in Eq. (3.5), and making use of Eq. (3.2), we obtain

$$f_{ij}[\partial u_p(t-\tau)/\partial X_q \Big|_{\tau=0}^{\infty} ; \partial x_p/\partial X_q] = a_{ki}(t)a_{\ell j}(t)f_{k\ell}[\partial \bar{u}_p(t-\tau)/\partial X_q \Big|_{\tau=0}^{\infty} ; \partial \bar{x}_p/\partial X_q] . \tag{6.3}$$

At this point, neither $\bar{x}_i$ nor $\bar{u}_i(t)$ have been defined. Only their sum, $\bar{x}_i(t)$, has been defined by (3.1). The new total deformation $\bar{x}_i(t)$ can be decomposed into a steady part and an unsteady part in either of two essentially different ways:

(i) the steady part $\bar{x}_i$ may be regarded as identical with the steady part x_i of the original deformation, in which case the entire rigid motion is included in the unsteady part $\bar{u}_i(t)$,

(ii) the new steady part $\bar{x}_i$ may be related to x_i by a time-independent rigid rotation, with $\bar{u}_i(t)$ related to $u_i(t)$ by the same rotation.

In case (i), we take $\bar{x}_i = x_i$. The displacement field $\bar{u}_i(t)$ is then defined by

$$\bar{u}_i(t) = \bar{x}_i(t) - \bar{x}_i = a_{ij}(t)x_j(t) - x_i \; . \tag{6.4}$$

Equation (6.3) then becomes

$$f_{ij}[\partial u_p \overset{\infty}{\underset{\tau=0}{(t-\tau)}}/\partial X_q \; ; \; \partial x_p/\partial X_q]$$

$$= a_{ki}(t)a_{\ell j}(t)f_{k\ell}[a_{pr}(t-\tau)\partial x_r \overset{\infty}{\underset{\tau=0}{(t-\tau)}}/\partial X_q - \partial x_p/\partial X_q \; ; \; \partial x_p/\partial X_q] . \tag{6.5}$$

By constructing the coefficients $a_{ij}(t)$ from the gradients $\partial x_i(t)/\partial X_j$, as in Section 3, we find that f_{ij} must be expressible in the form

$$f_{ij} = \frac{\partial x_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_\ell} F^*_{k\ell}[G_{pq} \overset{\infty}{\underset{\tau=0}{(t-\tau)}} \; ; \; \partial x_p/\partial X_q] \; . \tag{6.6}$$

When Eq. (6.6) is used in the basic invariance requirement (6.3), and Eqs. (3.1) and (3.2) are used to simplify the resulting expression, it is found that the functionals F^*_{ij} must satisfy the condition

$$F^*_{ij}[G_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}} \; ; \; \partial x_p/\partial X_q] = F^*_{ij}[\overline{G}_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}} \; ; \; \partial \overline{x}_p/\partial X_q] \; , \qquad (6.7)$$

where the strain components $\overline{G}_{pq}(t)$ are defined by replacing $x_i(t)$ by $\overline{x}_i(t)$ in Eq. (3.8). By making use of Eqs. (3.1) and (3.2), it is found that

$$\overline{G}_{pq}(t) = G_{pq}(t) \; . \qquad (6.8)$$

We now consider case (ii). The coefficients a_{ij} in Eqs. (3.1) and (3.2) are taken to be constants, and the new steady deformation $\overline{x}_i$ is defined by

$$\overline{x}_i = a_{ij}x_j \; . \qquad (6.9)$$

By using Eqs. (6.8) and (6.9) in Eq. (6.7), we find that the functionals F^*_{ij} must be of such form that

$$F^*_{ij}[G_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}} \; ; \; \partial x_p/\partial X_q] = F^*_{ij}[G_{pq}\overset{\infty}{\underset{\tau=0}{(t-\tau)}} \; ; \; a_{pr}\partial x_r/\partial X_q] \; . \qquad (6.10)$$

By constructing the coefficients a_{ij} from the steady deformation gradients $\partial x_p/\partial X_q$ in the manner described in Section 3, we find that the functionals F^*_{ij} depend on the gradients $\partial x_p/\partial X_q$ only through the strain components G_{pq} defined in terms of them. By using this result in Eq. (6.6), and the result so obtained in Eq. (6.1), we obtain finally

$$\sigma_{ij}(t) = \frac{\partial x_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_\ell} F_{k\ell}\left[\underset{\tau=0}{\overset{\infty}{G_{pq}}}(t-\tau) \; ; \; G_{pq} \right] . \qquad (6.11)$$

The displacement gradients $\partial u_i(t)/\partial X_j$ enter into Eq.(6.11) through $\partial x_i(t)/\partial X_j$ and $G_{pq}(t-\tau)$. In general, the displacements $u_i(t-\tau)$ cannot be small throughout the entire history of deformation, since $x_i + u_i(t-\tau) = X_i$ for τ sufficiently large. Thus, before Eq. (6.11) can be linearized with respect to the displacement gradients, it is necessery to assume that the material has a fading memory and that enough time has elapsed so that the undeformed state has been forgotten. With these assumptions, Eq. (6.11) can be linearized, and the form obtained is again given by Eq. (5.4).

7. Symmetry of the material

Certain restrictions must be placed on the functions F_{ij}, $A^{(v)}_{ijk\ell}$, and $K_{ijk\ell}$ in (5.4), if the constitutive equation is to describe a material which has some symmetry in its undeformed state. Let $\bar{x}$ be a fixed rectangular Cartesian coordinate system, related to the system x by an orthogonal transformation, so that

$$\bar{x}_i = s_{ij}x_j \,, \qquad \bar{X}_i = s_{ij}X_j \,, \qquad \bar{u}_i(t) = s_{ij}u_j(t), \quad (7.1)$$

where

$$s_{ik}s_{jk} = s_{ki}s_{kj} = \delta_{ij} \,. \quad (7.2)$$

Defining $\bar{G}_{pq}$, $\bar{E}_{pq}(t)$, and $\bar{E}^{(v)}_{pq}(t)$ as in Eqs. (3.8), (4.5), and (4.9) respectively, but in terms of the new coordinates $\bar{x}_i$ and $\bar{X}_i$, and making use of Eqs. (7.1) and (7.2), we obtain

$$\bar{G}_{ij} = s_{ik}s_{j\ell}G_{k\ell} \,, \qquad \bar{E}^{(v)}_{ij}(t) = s_{ik}s_{j\ell}E^{(v)}_{k\ell}(t). \quad (7.3)$$

The components $\bar{\sigma}_{ij}(t)$ of stress, measured with respect to the system $\bar{x}$, are given by

$$\bar{\sigma}_{ij}(t) = s_{ik}s_{j\ell}\sigma_{k\ell}(t) \,. \quad (7.4)$$

If $\underset{\sim}{s} = \|s_{ij}\|$ is a symmetry transformation for the material, then Eq. (5.4) remains valid if all quantities are measured with respect to the system $\bar{x}$ instead of the system x. By using Eq. (5.4) and the corresponding expression for $\bar{\sigma}_{ij}(t)$ to eliminate $\sigma_{ij}(t)$ and $\bar{\sigma}_{ij}(t)$ from Eq. (7.4), a relation involving F_{ij}, $A^{(v)}_{ijk\ell}$, and $K_{ijk\ell}$ is obtained. From that relation and Eqs. (7.1) to (7.3), it can be shown after some manipulation that the following conditions must be satisfied (for details of a similar manipulation in the isotropic case, see [1] or [2]):

$$F_{ij}(\overline{G}_{pq}) = s_{ik}s_{j\ell}F_{k\ell}(G_{pq}) , \qquad (7.5)$$

$$A^{(v)}_{ijrs}(\overline{G}_{pq})\overline{E}^{(v)}_{rs}(t) = s_{ik}s_{j\ell}A^{(v)}_{k\ell rs}(G_{pq})E^{(v)}_{rs}(t) , \qquad (7.6)$$

$$K_{ijrs}(\overline{G}_{pq},\tau)\overline{E}_{rs}(t-\tau) = s_{ik}s_{j\ell}K_{k\ell rs}(G_{pq},\tau)E_{rs}(t-\tau). \qquad (7.7)$$

These relations must be satisfied identically for each transformation $\underset{\sim}{s}$ belonging to the group of symmetries of the material. In the following section we will consider the consequences of Eqs. (7.5) to (7.7) in the case of isotropic materials.

8. Isotropic materials

The symmetry group for an isotropic material is either the full orthogonal group or the proper orthogonal group, depending on whether the material has a center of symmetry or not. Presence or absence of a center of symmetry is irrelevant in the present context, however, because Eqs. (7.5) to (7.7) are satisfied identically if the transformation is the central inversion, i.e. if $s_{ij} = -\delta_{ij}$.

Rivlin and Ericksen [4] have studied the implications of an equation of the form (7.5). They have shown that if the functions F_{ij} are single-valued functions of the variables G_{pq}, and if Eq. (7.5) is satisfied for every orthogonal transformation $\underset{\sim}{s}$, then the matrix $\underset{\sim}{F} = \|F_{ij}\|$ can be expressed in terms of the matrix $\underset{\sim}{G} = \|G_{ij}\|$ in the form

$$\underset{\sim}{F} = F_0\underset{\sim}{I} + F_1\underset{\sim}{G} + F_2\underset{\sim}{G}^2. \qquad (8.1)$$

Here $\underset{\sim}{I}$ is the unit matrix $\|\delta_{ij}\|$. The scalar coefficients F_0, F_1 and F_2 are single-valued functions of three basic orthogonal invariants of $\underset{\sim}{G}$,

$$\operatorname{tr} \underset{\sim}{G} = G_{ii}, \qquad \operatorname{tr} \underset{\sim}{G}^2 = G_{ij}G_{ij}, \qquad \operatorname{tr} \underset{\sim}{G}^3 = G_{ij}G_{jk}G_{ki} \,. \qquad (8.2)$$

If, in (7.5), F_{ij} is a polynomial* in its arguments, then $\underset{\sim}{F}$ may be expressed in the form (8.1), with F_0, F_1, and F_2 expressible as polynomials in the invariants (8.2).

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* We note that if F_{ij} is a continuous function of G_{pq}, it may be expressed as a polynomial as closely as desired in any closed domain.

The limitations on $K_{ijrs}E_{rs}(t-\tau)$ implied by (7.7) may be obtained in more explicit form by considerations of the type used by Rivlin and Ericksen [4] and by Rivlin [5]. We use the notation

$$K_{ij}(G_{pq},E_{rs}) = K_{ijrs}(G_{pq})E_{rs} \, , \tag{8.3}$$

the dependence of E_{rs} on $t-\tau$ being understood. Then, Eq. (7.7) may be written in the form

$$K_{ij}(\bar{G}_{pq},\bar{E}_{rs}) = s_{ik}s_{j\ell}K_{k\ell}(G_{pq},E_{rs}) \, . \tag{8.4}$$

If we assume that K_{ijrs} is a polynomial in G_{pq}, so that K_{ij} is a polynomial in G_{pq} and E_{rs}, linear in E_{rs}, it follows (Rivlin [5]) that $\underset{\sim}{K} = \|K_{ij}\|$ may be expressed in the form

$$\begin{aligned} \underset{\sim}{K} = {} & K_o\underset{\sim}{E} + K_1(\underset{\sim}{E}\underset{\sim}{G} + \underset{\sim}{G}\underset{\sim}{E}) + K_2(\underset{\sim}{E}\underset{\sim}{G}^2 + \underset{\sim}{G}^2\underset{\sim}{E}) \\ & + \sum_{M,N=0}^{2} K_{MN}(\operatorname{tr} \underset{\sim}{E}\underset{\sim}{G}^N)\underset{\sim}{G}^M \, . \end{aligned} \tag{8.5}$$

The coefficients K_o, K_1, K_2 and K_{MN} are polynomials in the invariants (8.2), and functions of τ. If these coefficients are regarded as single-valued functions rather than polynomials, Eq. (8.4) is still satisfied.

Since Eq. (7.6) is of the same form as Eq. (7.7), its implications are analogous to those of (7.7). We use the notation

$$A^{(v)}_{ij}(G_{pq},E^{(v)}_{rs}) = A^{(v)}_{ijrs}(G_{pq})E^{(v)}_{rs}(t) \, . \tag{8.6}$$

By an argument similar to that used in deriving (8.5), we find that $\underset{\sim}{A}^{(v)} = \|A^{(v)}_{ij}\|$ can be expressed in the form

$$\underset{\sim}{A}^{(v)} = A_o^{(v)}\underset{\sim}{E}^{(v)} + A_1^{(v)}(\underset{\sim}{E}^{(v)}\underset{\sim}{G}+\underset{\sim}{G}\underset{\sim}{E}^{(v)}) + A_2^{(v)}(\underset{\sim}{E}^{(v)}\underset{\sim}{G}^2+\underset{\sim}{G}^2\underset{\sim}{E}^{(v)})$$

$$+ \sum_{M,N=0}^{2} A_{MN}^{(v)}(\mathrm{tr}\ \underset{\sim}{E}^{(v)}\underset{\sim}{G}^N)\underset{\sim}{G}^M . \tag{8.7}$$

The coefficients $A_o^{(v)}$, $A_1^{(v)}$, $A_2^{(v)}$, and $A_{MN}^{(v)}$ are polynomials in the invariants (8.2). Again, the invariance requirements are still satisfied if these coefficients are regarded as single-valued functions rather than polynomials.

When Eq. (8.5) is used in (8.3), (8.7) is used in (8.6), and (8.1), (8.3), and (8.6) are then used in Eq. (5.4), the constitutive equation for isotropic materials is obtained. It may be written as

$$\sigma_{ij}(t)$$

$$= \left[\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_\ell} + \frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j}{\partial X_\ell} + \frac{\partial x_i}{\partial X_k}\frac{\partial u_j(t)}{\partial X_\ell}\right][F_o\delta_{k\ell} + F_1G_{k\ell} + F_2G_{kp}G_{p\ell}]$$

$$+ \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_\ell}\sum_v [A_o^{(v)}\delta_{kp}\delta_{\ell q} + A_1^{(v)}(\delta_{kp}G_{\ell q} + \delta_{\ell q}G_{kp})$$

$$+ A_2^{(v)}(\delta_{kp}G_{\ell r}G_{rq} + \delta_{\ell q}G_{kr}G_{rp})$$

$$+ \sum_{M,N=0}^{2} A_{MN}^{(v)}(\underset{\sim}{G}^M)_{k\ell}(\underset{\sim}{G}^N)_{pq}]E_{pq}^{(v)}(t)$$

$$+ \frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_\ell}\int_0^\infty [K_o(\tau)\delta_{kp}\delta_{\ell q} + K_1(\tau)(\delta_{kp}G_{\ell q} + \delta_{\ell q}G_{kp})$$

$$+ K_2(\tau)(\delta_{kp}G_{\ell r}G_{rq} + \delta_{\ell q}G_{kr}G_{rp})$$

$$+ \sum_{M,N=0}^{2} K_{MN}(\tau)(\underset{\sim}{G}^M)_{k\ell}(\underset{\sim}{G}^N)_{pq}]E_{pq}(t-\tau)d\tau. \tag{8.8}$$

Dependence of the scalar coefficients on tr $\underset{\sim}{G}^N$ (N=1,2,3) is understood.

9. An alternative form for the constitutive equation for isotropic materials

Equation (8.8) can be written in a slightly simpler form. As the first step, the displacement gradients are written in terms of spatial derivatives, thus

$$\frac{\partial u_i(t)}{\partial X_j} = \frac{\partial u_i(t)}{\partial x_k}\frac{\partial x_k}{\partial X_j} . \tag{9.1}$$

Here the derivatives $\partial x_k/\partial X_j$ are deformation gradients for the basic steady deformation. By using (9.1) in the definition (4.5) of E_{pq}, we obtain

$$E_{pq}(t) = \frac{\partial x_i}{\partial X_p}\frac{\partial x_j}{\partial X_q} e_{ij}(t) , \tag{9.2}$$

where e_{ij} is the classical strain tensor defined by

$$e_{ij}(t) = \tfrac{1}{2}\left[\frac{\partial u_i(t)}{\partial x_j} + \frac{\partial u_j(t)}{\partial x_i}\right] . \tag{9.3}$$

The time-derivatives $E_{pq}^{(\mu)}$ are, from (9.2),

$$E_{pq}^{(\mu)}(t) = \frac{\partial x_i}{\partial X_p}\frac{\partial x_j}{\partial X_q} e_{ij}^{(\mu)}(t) , \tag{9.4}$$

where

$$e_{ij}^{(\mu)}(t) = d^{\mu}e_{ij}(t)/dt^{\mu} . \tag{9.5}$$

In the definition (4.9) of $E_{pq}^{(\mu)}$, and now in Eq. (9.5), the differentiation d/dt holds X_p fixed. Thus

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial u_i(t)}{\partial t}\frac{\partial}{\partial x_i} . \tag{9.6}$$

When this expression is used in (9.5), terms arise which are quadratic in the displacements. These will be neglected. Thus, in (9.5), d/dt may be regarded as the partial derivative with x_i held fixed.

Next, consider the tensor $\underset{\sim}{g} = \|g_{ij}\|$, defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_p} \frac{\partial x_j}{\partial X_p} . \tag{9.7}$$

By using this definition with Eqs. (9.2), (9.4), and the definition (3.8) of G_{pq}, we obtain the relations

$$\text{tr } \underset{\sim}{G}^N = \text{tr } \underset{\sim}{g}^N , \quad \text{tr}(\underset{\sim}{E}^{(v)}\underset{\sim}{G}^N) = \text{tr}(\underset{\sim}{e}^{(v)}\underset{\sim}{g}^{N+1}), \tag{9.8}$$

$$\frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_\ell} (\underset{\sim}{G}^N)_{k\ell} = (\underset{\sim}{g}^{N+1})_{ij} \tag{9.9}$$

$$\frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_\ell}(\underset{\sim}{E}^{(v)}\underset{\sim}{G}^N)_{k\ell} = (\underset{\sim\sim}{ge}^{(v)}\underset{\sim}{g}^{N+1})_{ij} , \tag{9.10}$$

$$\left[\frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_\ell} + \frac{\partial u_i(t)}{\partial X_k} \frac{\partial x_j}{\partial X_\ell} + \frac{\partial x_i}{\partial X_k} \frac{\partial u_j(t)}{\partial X_\ell}\right](\underset{\sim}{G}^N)_{k\ell}$$
$$= [\delta_{ik}\delta_{j\ell} + u_{i,k}(t)\delta_{j\ell} + u_{j,\ell}(t)\delta_{ik}](\underset{\sim}{g}^{N+1})_{k\ell}. \tag{9.11}$$

In the last equation, the notation $\partial u_i/\partial x_j = u_{i,j}$ has been used.

By making use of Eqs. (9.8) to (9.11), the constitutive equation (8.8) can be written in the form

$$\begin{aligned}
\sigma_{ij}(t) = {} & [\delta_{ik}\delta_{j\ell} + u_{i,k}(t)\delta_{j\ell} + u_{j,\ell}(t)\delta_{ik}](F_o\underset{\sim}{g} + F_1\underset{\sim}{g}^2 + F_2\underset{\sim}{g}^3)_{k\ell} \\
& + \sum_v [A_o^{(v)}\underset{\sim\sim}{ge}^{(v)}\underset{\sim}{g} + A_1^{(v)}(\underset{\sim\sim}{ge}^{(v)}\underset{\sim}{g}^2 + \underset{\sim}{g}^2\underset{\sim}{e}^{(v)}\underset{\sim}{g}) \\
& + A_2^{(v)}(\underset{\sim\sim}{ge}^{(v)}\underset{\sim}{g}^3 + \underset{\sim}{g}^3\underset{\sim}{e}^{(v)}\underset{\sim}{g}) + \sum_{M,N=0}^{2} A_{MN}^{(v)} \text{tr}(\underset{\sim}{e}^{(v)}\underset{\sim}{g}^{N+1})\underset{\sim}{g}^{M+1}]_{ij} \\
& + \int_0^\infty [K_o(\tau)\underset{\sim\sim\sim}{geg} + K_1(\tau)[\underset{\sim\sim\sim}{geg}^2+\underset{\sim}{g}^2\underset{\sim\sim}{eg}) + K_2(\tau)(\underset{\sim\sim\sim}{geg}^3+\underset{\sim}{g}^3\underset{\sim\sim}{eg}) \\
& \qquad + \sum_{M,N=0}^{2} K_{MN}(\tau)\text{tr}(\underset{\sim\sim}{eg}^{N+1})\underset{\sim}{g}^{M+1}]_{ij}d\tau.
\end{aligned} \tag{9.12}$$

The time-dependence of $\underset{\sim}{e}$ and $\underset{\sim}{e}^{(v)}$ is understood. According to (9.8), the scalar coefficients in (9.12) may be regarded as functions of tr $\underset{\sim}{g}^N$ (N=1,2,3) rather than of tr $\underset{\sim}{G}^N$ (N=1,2,3).

The integration with respect to time in (9.12) is integration with X_p held fixed. However, with neglect of terms quadratic in the displacements, it may be regarded as an integration holding x_i fixed.

By using the Cayley-Hamilton theorem, $\underset{\sim}{g}^3$ may be expressed as a linear combination of $\underset{\sim}{g}^2$, $\underset{\sim}{g}$, and $\underset{\sim}{I}$, with coefficients which are polynomials in the invariants tr $\underset{\sim}{g}^N$ (N=1,2,3). Therefore we may write

$$F_0\underset{\sim}{g} + F_1\underset{\sim}{g}^2 + F_2\underset{\sim}{g}^3 = f_0\underset{\sim}{I} + f_1\underset{\sim}{g} + f_2\underset{\sim}{g}^2 \ , \qquad (9.13)$$

where f_0, f_1, and f_2 are functions of tr $\underset{\sim}{g}^N$ (N=1,2,3).

Denoting the integrand in Eq. (9.12) by k_{ij} and its matrix by $\underset{\sim}{k} = \|k_{ij}\|$, we see that $\underset{\sim}{k}$ is a symmetric matrix polynomial in the symmetric matrices $\underset{\sim}{g}$ and $\underset{\sim}{e}$, linear in $\underset{\sim}{e}$. The scalar coefficients are, of course, not necessarily polynomials. According to Rivlin [5], $\underset{\sim}{k}$ may therefore be expressed in a form similar to (8.5). In that expression we replace $\underset{\sim}{K}$ by $\underset{\sim}{k}$, $\underset{\sim}{G}$ by $\underset{\sim}{g}$ and $\underset{\sim}{E}$ by $\underset{\sim}{e}$ to get the desired result. Similarly, the summand in (9.12) can be expressed in a form analogous to Eq. (8.7), by replacing $\underset{\sim}{G}$ by $\underset{\sim}{g}$ and $\underset{\sim}{E}$ by $\underset{\sim}{e}$ in the latter expression. These results, taken with Eq.(9.13), imply that Eq. (9.12) can be reduced to the following form:

$$\begin{aligned}
\sigma_{ij}(t) = {} & [\delta_{ik}\delta_{j\ell} + u_{i,k}(t)\delta_{j\ell} + u_{j,\ell}(t)\delta_{ik}][f_0\underset{\sim}{I} + f_1\underset{\sim}{g} + f_2\underset{\sim}{g}^2]_{k\ell} \\
& + \sum_v [a_0^{(v)}\delta_{ik}\delta_{j\ell} + a_1^{(v)}(\delta_{ik}g_{j\ell}+\delta_{j\ell}g_{ik}) \\
& + a_2^{(v)}(\delta_{ik}g_{jn}g_{n\ell}+\delta_{j\ell}g_{in}g_{nk}) \\
& + \sum_{M,N=0}^{2} a_{MN}^{(v)}(\underset{\sim}{g}^M)_{ij}(\underset{\sim}{g}^N)_{k\ell}]e_{k\ell}^{(v)}(t) \\
& + \int_0^\infty [k_0(\tau)\delta_{ik}\delta_{j\ell} + k_1(\tau)(\delta_{ik}g_{j\ell}+\delta_{j\ell}g_{ik}) \\
& + k_2(\tau)(\delta_{ik}g_{jn}g_{n\ell}+\delta_{j\ell}g_{in}g_{nk}) \\
& + \sum_{M,N=0}^{2} k_{MN}(\tau)(\underset{\sim}{g}^M)_{ij}(\underset{\sim}{g}^N)_{k\ell}]e_{k\ell}(t-\tau)d\tau. \qquad (9.14)
\end{aligned}$$

The scalar coefficients are functions of **tr** $\underset{\sim}{g}^N$ (N=1,2,3).

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